Computing maximal subsemigroups

THE CONCLUSION

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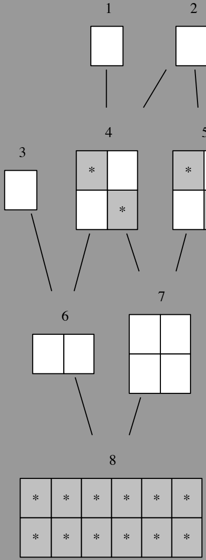
What is a semigroup?



It's a set with an associative binary operation, for example

- $(\varnothing, \varnothing)$.
- $(\mathbb{N}, +)$.

A visual interpretation is useful.

What is a maximal subsemigroup?

Today, maximality is all about *containment*.

Definition (Maximal subsemigroup)

A maximal subsemigroup is a proper subsemigroup which is not contained in any other proper subsemigroup.

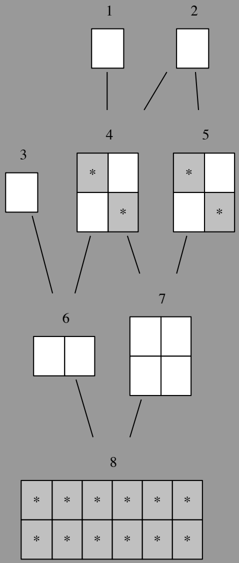
Examples of semigroups and their maximal subsemigroups

- $\varnothing$ is a maximal subsemigroup of the trivial semigroup.
- $\mathbb{R} \setminus \{0\}$ is a maximal subsemigroup of $(\mathbb{R}, \times)$.
- $\mathbb{R}^+$ is a maximal subsemigroup of $(\mathbb{R} \setminus \{0\}, \times)$.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is a maximal subsemigroup of $(\mathbb{Z}, +)$.
- $\{0, 2, 3, \ldots\}$ is a maximal subsemigroup of $(\mathbb{N}, +)$.
- $\{1, 2, 3, \ldots\}$ is a maximal subsemigroup of $(\mathbb{N}, +)$.
- $\{2, 3, \ldots\}$ is *the* maximal subsemigroup of $(\{1, 2, \ldots\}, +)$.
- $T_n \setminus \{$maps of rank $n-1\}$ is a maximal subsemigroup of T_n.

Some questions

- Does every semigroup have maximal subsemigroups?
  - $((1, \infty), \times)$?
- How many maximal subsemigroups can a semigroup have?
- How big is a maximal subsemigroup?
- Are maximal subsemigroups interesting?
- How do you calculate the maximal subsemigroups of a semigroup?

Results from Graham, Graham, and Rhodes



A maximal subsemigroup lacks part of precisely one $\mathscr{D}$-class.

The remaining part of it either:

1. is a union of rows and columns
2. is a union of only rows
3. is a union of only columns
4. contains part of *every* $\mathscr{H}$-class
5. is empty

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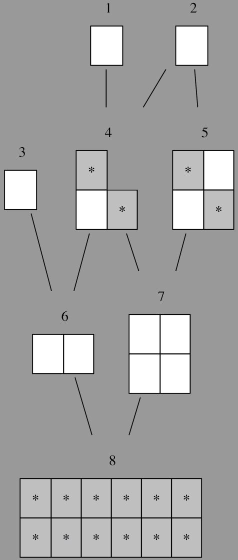

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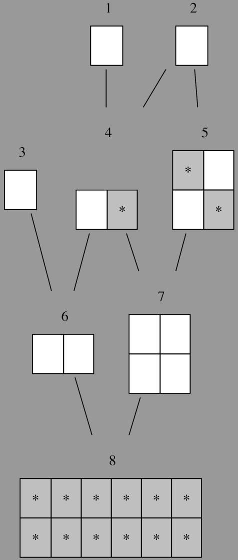

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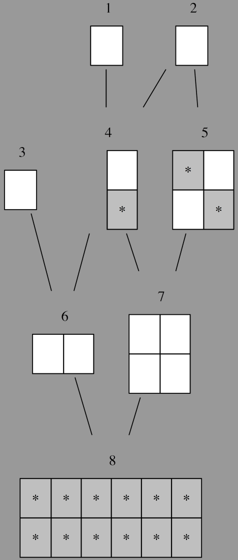

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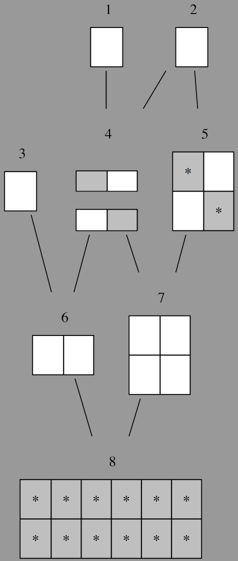

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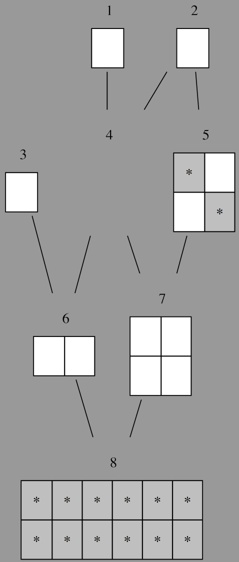

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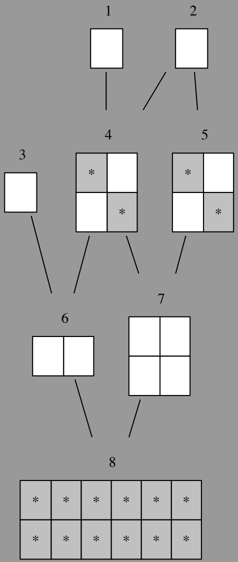
The overall technique



Go through each $\mathscr{D}$-class in turn.

Work out which maximal subsemigroups are possible of each type, if any.

The principal factor of a $\mathscr{D}$-class

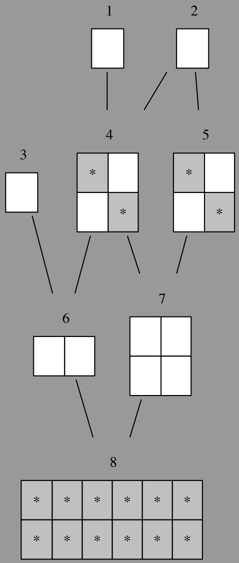
The *principal factor* of a $\mathscr{D}$-class D is a semigroup, denoted D^*.

It is defined to be $(D \cup \{0\}, *)$, with the operation $*$ defined by:

$$x * y = \begin{cases} xy & \text{if } x, y, xy \in D. \\ 0 & \text{otherwise.} \end{cases}$$

The principal factor is essentially a *Rees 0-matrix semigroup*.

Some of the problems to consider



Foremost, a maximal subsemigroup must be a **subsemigroup**.

So, a maximal subsemigroup must:

- give a subsemigroup of the principal factor.
- contain everything generated by the other $\mathscr{D}$-classes.
- be closed under the action of the other $\mathscr{D}$-classes.

Understanding these problems with digraphs.

For (most of) the rest of this talk, D is a *specific* $\mathscr{D}$-class.

We can't consider D in isolation since the rest of the semigroup interferes.

We construct *three* digraphs relative to D, and use them to calculate the maximal subsemigroups which you can get by removing bits of D.

The digraphs $\Gamma_{\mathscr{R}}$ and $\Gamma_{\mathscr{L}}$.

$\Gamma_{\mathscr{R}}$ captures:

- how the rest of semigroup moves around the **rows** of D;
- what the rest of the semigroup generates inside D.

Time for an example on the board.

$\Gamma_{\mathscr{L}}$ is the 'dual' of this, for columns of D.

The graph Γ.

Γ captures:

- how the rest of the semigroup moves around the **idempotents** of D;
- what the rest of the semigroup generates in D.

Time for an example on the board.

Maximal subsemigroups of type (1)

Let M be a subset such that $M \cap D$ is a union of rows and columns.

Theorem

M is a maximal subsemigroup if and only if

- *the rows form a collection of strongly connected components of* $\Gamma_{\mathscr{R}}$;
- *the cols form a collection of strongly connected components of* $\Gamma_{\mathscr{L}}$;
- *these collections each have no out-neighbours in* $\Gamma_{\mathscr{R}}$ *or* $\Gamma_{\mathscr{L}}$;
- *every red edge of* Γ *is incident to either a row or column in* M;
- *the subgraph of* Γ *induced by the rows and columns of* M *is maximal with respect to containing no black edges.*

Maximal subsemigroups of types (2) and (3)

Let M be a subset such that $M \cap D$ is a union of rows only.

Theorem

M is a maximal subsemigroup if and only if:

- *M is not contained in a maximal subsemigroup of type (1);*
- *the missing rows form a strongly connected component of $\Gamma_{\mathscr{R}}$;*
- *that strongly connected component has no in-neighbours in $\Gamma_{\mathscr{R}}$;*
- *that strongly connected component is not red.*

The theorem for maximal subsemigroups of type (3) is dual.

Maximal subsemigroups of type (4)

- Define $E(D)$ to be the set of idempotents of D.
- Define X to be the set of generators *above* D.

Let M be a subset which intersects every $\mathscr{H}$-class of D.

Theorem

M is a maximal subsemigroup if and only if $(M \cap D) \cup \{0\}$ is a maximal subsemigroup of D^* containing $E(D)X \cap D$.

Maximal subsemigroups of type (5)

Theorem

A maximal subsemigroup can be formed by removing D if and only if there are no maximal subsemigroups of types (1) to (4), and D isn't generated by the rest of the semigroup.

Summary of the algorithm.

Work out all of the information contained in the semigroup diagram:

- the Green's relations ($\mathscr{D}$, $\mathscr{R}$, and $\mathscr{L}$);
- the $\mathscr{D}$-class partial order;
- the location of the generators;
- the idempotents.

Go through each $\mathscr{D}$-class in turn:

- Construct $\Gamma_{\mathscr{R}}$, $\Gamma_{\mathscr{L}}$, and Γ;
- Use these digraphs to find maximal subsemigroups of types (1) to (3);
- Look for maximal subsemigroups of type (4);
- Look for maximal subsemigroups of type (5).

Return the results!

End.