Computing maximal subsemigroups

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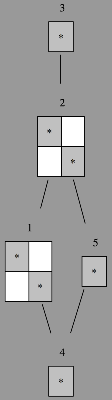
Semigroup diagrams

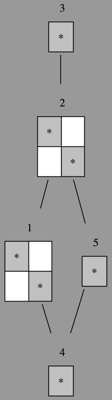
Green's relations $\mathscr{D}, \mathscr{L}, \mathscr{R}, \mathscr{H}$ partition a semigroup in a useful way.



$\mathscr{D}$-classes	the blocks
$\mathscr{L}$-classes	the columns in a block
$\mathscr{R}$-classes	the rows in a block
$\mathscr{H}$-classes	the small squares

Graham, Graham, Rhodes

1968: Maximal subsemigroups of finite semigroups



A maximal subsemigroup:

- lacks part of one $\mathscr{D}$-class.

A maximal subsemigroup either:

- is a union of $\mathscr{H}$-classes, or
- intersects every $\mathscr{H}$-class.

The remainder of the $\mathscr{D}$-class has a special form.

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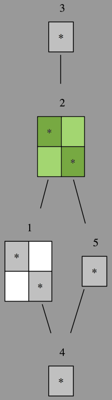

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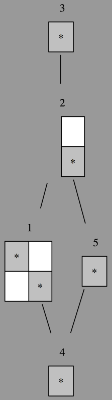

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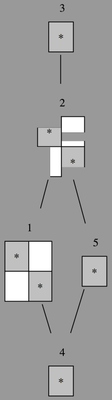

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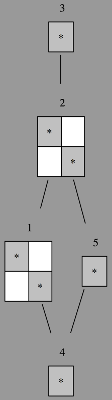

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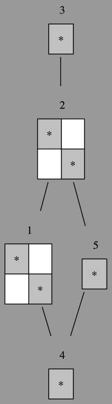
Maximal semigroups of arbitrary finite semigroups



Only need to consider $\mathscr{D}$-classes which contain generators.

Maximal $\mathscr{D}$-classes:

- Correspondence with maximal subsemigroups of a special Rees 0-matrix semigroup.

Non-maximal $\mathscr{D}$-class:

- Non-regular: remove entirely.
- Regular: more complicated.

Rees 0-matrix semigroups

- Let G be a semigroup (group).
- Let I and Λ be index sets.
- Let $P = (p_{\lambda,i})$ be a $|\Lambda| \times |I|$ matrix with entries in $G \cup \{0\}$.

Then the Rees 0-matrix semigroup $\mathscr{M}^0[I, G, \Lambda; P]$ is the set

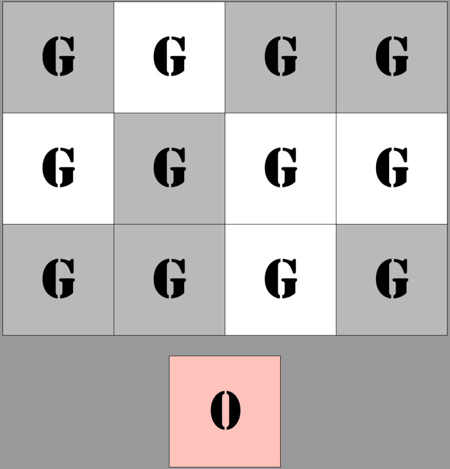
$$(I \times G \times \Lambda) \cup \{0\}$$

with multiplication

$$0x = x0 = 0 \text{ for all } x \in \mathscr{M}^0[I, G, \Lambda; P], \text{ and}$$

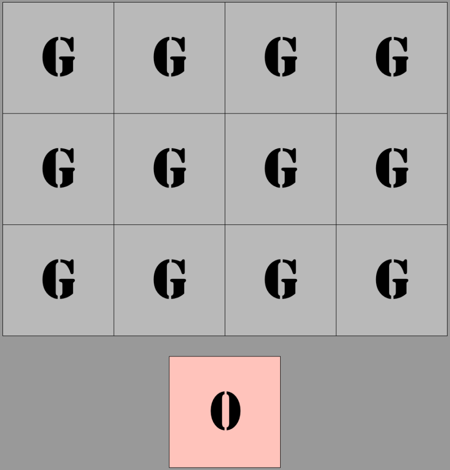
$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda,j}h, \mu) & \text{if } p_{\lambda,j} \neq 0, \\ 0 & \text{if } p_{\lambda,j} = 0. \end{cases}$$

Visualising a Rees 0-matrix semigroup



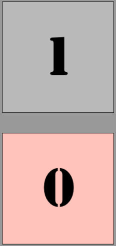
$$P = \begin{pmatrix} g_1 & 0 & g_2 \\ 0 & g_3 & g_4 \\ g_5 & 0 & 0 \\ g_6 & 0 & g_7 \end{pmatrix}$$

Remove the zero



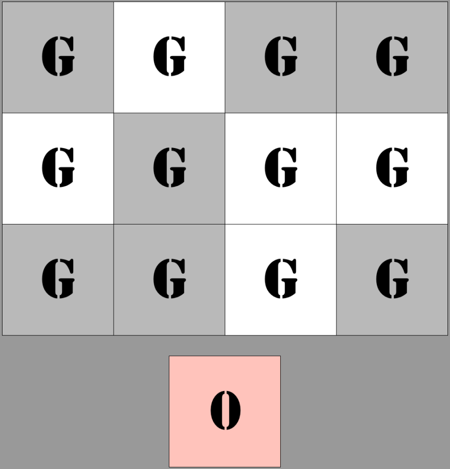
$$P = \begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \\ g_{10} & g_{11} & g_{12} \end{pmatrix}$$

Keep the zero



$$P = (g_1)$$

Remove a row



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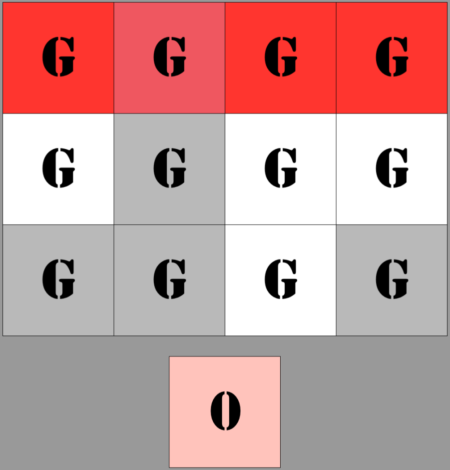

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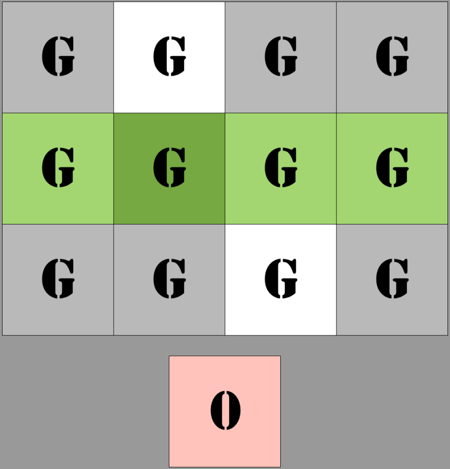

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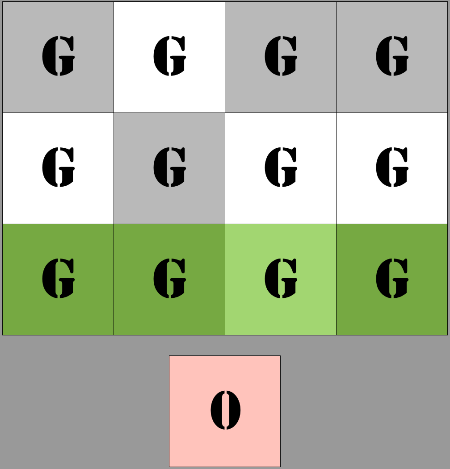

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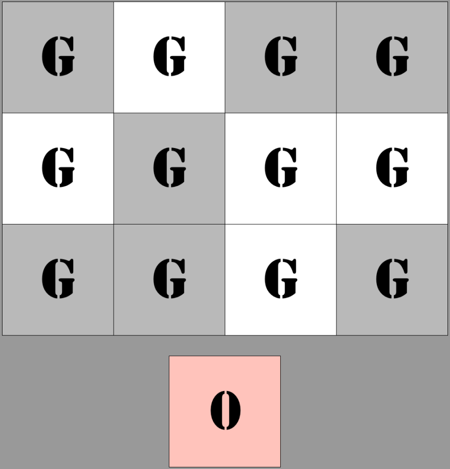

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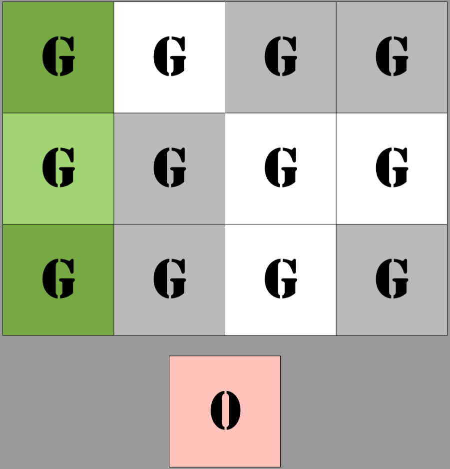

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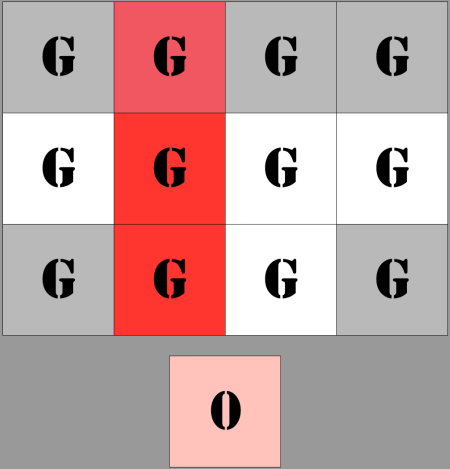

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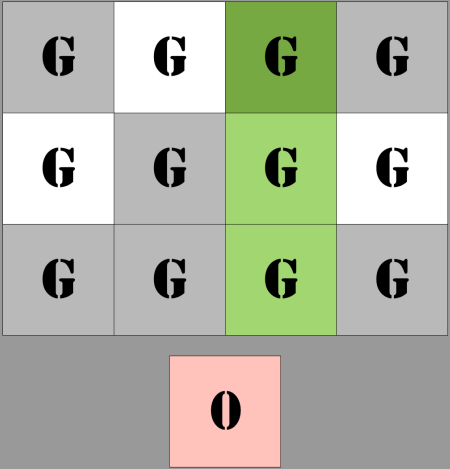

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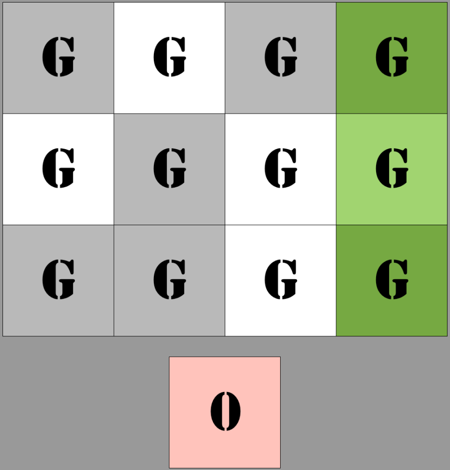

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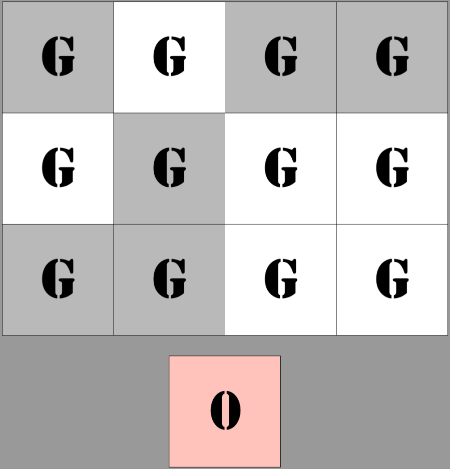

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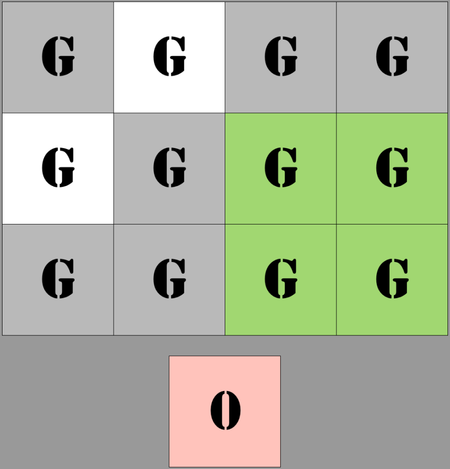

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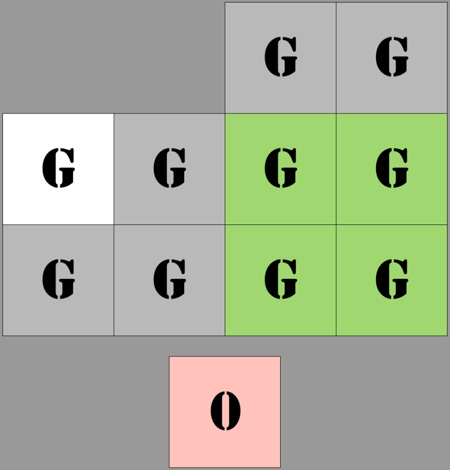

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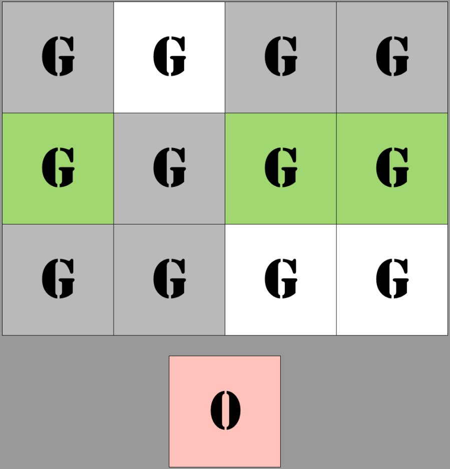

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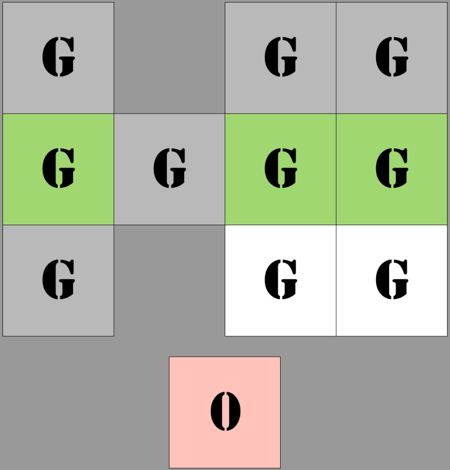

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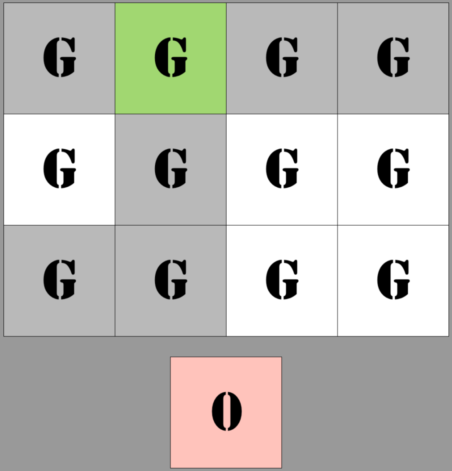

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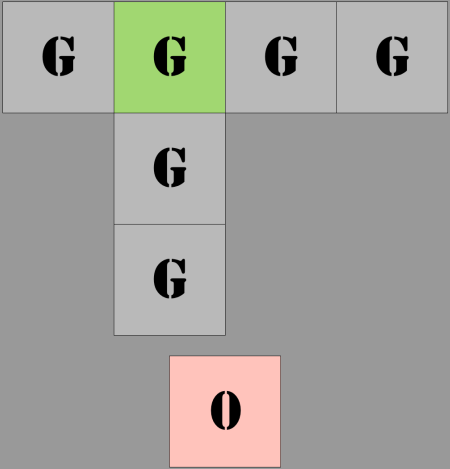

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The difficult case: "maximal subgroup" type

Let $R = \mathscr{M}^0[I, G, \Lambda; P]$ be a Rees 0-matrix semigroup.

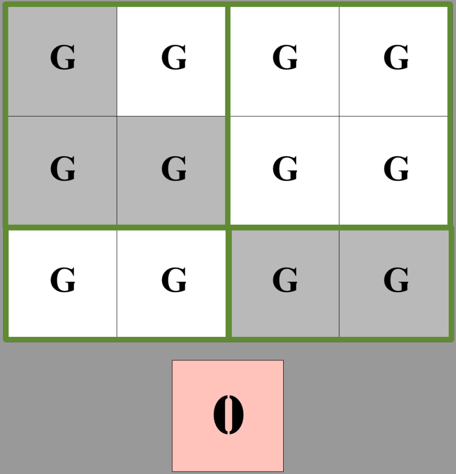
If $M \leq_{\text{max}} R$ intersects every $\mathscr{H}$-class of R non-trivially then

$$M \cong \mathscr{M}^0[I, V, \Lambda; Q],$$

where:

- $V \leq_{\text{max}} G$, and
- Q is a $|\Lambda| \times |I|$ matrix over $V \cup \{0\}$.

The difficulty



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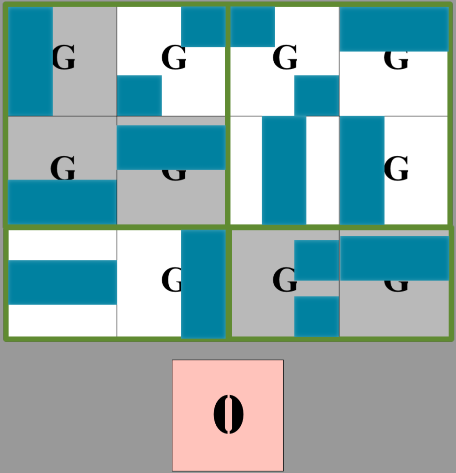


Different approaches

Old approach:

- These maximal subsemigroups have a specific type of generating set.
- Create all the generating sets, and discard those that generate R.

New approach:

- Perform an easy normalization.
- This way we obtain groups G_k (one for each connected component).
- Perform some easy group-theoretic calculations.

New approach

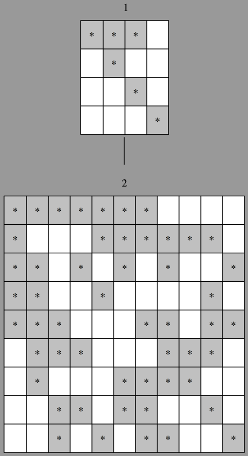
$g_1^{-1}Vg_1$	$g_1^{-1}Vg_1$	$g_1^{-1}Vg_2$	$g_1^{-1}Vg_2$
$g_1^{-1}Vg_1$	$g_1^{-1}Vg_1$	$g_1^{-1}Vg_2$	$g_1^{-1}Vg_2$
$g_2^{-1}Vg_1$	$g_2^{-1}Vg_1$	$g_2^{-1}Vg_2$	$g_2^{-1}Vg_2$

0

This is a maximal subsemigroup if and only if:

1. $V \leq_{\mathsf{max}} G$, and
2. $G_k \leq g_k^{-1} V g_k$ for all k.

Arbitrary finite semigroups: non-maximal regular $\mathscr{D}$-classes



Let M be a maximal subsemigroup arising from the lower $\mathscr{D}$-class, D.

Then either:

- M lacks some rows of D.
- M lacks some columns of D.
- $M \cap D$ corresponds to a maximal semigroup of the Rees 0-matrix semigroup of "maximal rectangle" or "maximal subgroup" type.

End.